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An Extension of the Transposition Theorems of Farkas and Eisenberg

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INTRODUCTION

In [3], Eisenberg shows that for any $A \in R^{m \times n}$, $b \in R^n$, $B \in R^{n \times n}$, B positive semidefinite symmetric, the system of equations and inequalities

$$Az \geqslant 0, \; z^t B z \leqslant 1, \; A^t y = b + Bz, \; y \geqslant 0 \tag{1}$$

is consistent, if and only if

$$Ax \geqslant 0 \text{ implies } (x^t B x)^{1/2} + b^t x \geqslant 0. \tag{2}$$

Setting $B = 0$ yields

$$A^t y = b, \qquad y \geqslant 0 \tag{3}$$

is consistent, if and only if

$$Ax \geqslant 0 \text{ implies } b^t x \geqslant 0. \tag{4}$$

(The requirement that $Az \geqslant 0$ may be dropped from (3) since it is always satisfied by $z = 0$. In fact, we will show that it can also be eliminated from (1), i.e., that Eisenberg's result holds with or without the restriction $Az \geqslant 0$.) The last result is the well-known Farkas Lemma [4], that is useful in establishing the duality theorems of mathematical programming (see e.g ([8], p. 18)).

Here we extend Eisenberg's result to convex polyhedral cones in complex space, thus obtaining generalizations of results of Levinson [7], Kaul [6], Mehndiratta [9] and Ben-Israel [2], in addition to [3] and [4].

Notation and Definitions

F — a field, here either

R — the real field or

C — the complex field,

F^n — the n-dimensional vector space over F

$F^{m\times n}$ — the $m \times n$ matrices over F.

$R_+{}^n = \{x \in R^n, x_i \geqslant 0\ (i = 1,..., n)\}$, the non-negative orthant of R^n. For any $x, y \in R^n$, $x \geqslant y$ denotes $x - y \in R_+{}^n$. $S \subset C^n$ is a polyhedral convex cone if there is a positive integer K and a matrix $A \in C^{n\times K}$ such that

$$S = AR_+{}^K = \{Ax : x \geqslant 0\}.$$

(An alternate, and equivalent definition, S is a polyhedral convex cone if it is the nonempty intersection of finitely many closed half spaces, each having 0 in its boundary.) The polar of $S \subset C^n$ is defined by

$$S^* = \{y \in C^n, x \in S \text{ implies } \mathrm{Re}(x^*y) \geqslant 0\}.$$

For a listing of some of the properties and examples of polyhedral convex cones and their polars that will be pertinent here, see [2]. Superscript t denotes transpose, superscript $*$, when applied to matrices and vectors, will denote conjugate transpose.

0 will denote matrices or vectors of appropriate dimension with 0 in every position. The meaning and dimension will be clear from the context.

Results

We list a number of previously established results that will be needed in proving our main theorem (Theorem 4).

Lemma 1. *Let $B \in C^{n\times n}$ be positive semidefinite hermitian. Then*

$$\mathrm{Re}\, x^*Bz \leqslant (x^*Bx)^{1/2}\,(z^*Bz)^{1/2}. \tag{5}$$

Proof. From the generalized Schwarz inequality ([11], p. 262),

$$|\, x^*Bz \,| \leqslant (x^*Bx)^{1/2}\,(z^*Bz)^{1/2},$$

(5) follows, since $\mathrm{Re}\, x^*Bz \leqslant |\, x^*Bz \,|$.

THEOREM 1 [1]. *Let $A \in C^{m\times n}$, $b \in C^m$, $c \in C^n$ and $S \subset C^n$, $T \subset C^m$ be polyhedral convex cones. Consider the pair of problems:*

$$\text{P1.} \qquad \begin{aligned} &\text{Minimize } f(x) \equiv \operatorname{Re}[\tfrac{1}{2}x^*Bx + c^*x] \\ &\text{Subject to } Ax - b \in T \\ &\qquad\qquad\quad x \in S \end{aligned}$$

$$\text{P2.} \qquad \begin{aligned} &\text{Maximize } g(y, z) \equiv \operatorname{Re}(-\tfrac{1}{2}y^*By + b^*z) \\ &\text{Subject to } -A^*z + By + c \in S^* \\ &\qquad\qquad\qquad\qquad\quad z \in T^*. \end{aligned}$$

If P1 *has an optimal solution x_o, then there exists a vector z_o such that (x_o, z_o) is an optimal solution of* P2 *and $f(x_o) = g(y_o, z_o)$.*

THEOREM 2 [2]. *Let $A \in C^{m\times n}$, $b \in C^m$ and let $S \subset C^n$ be a polyhedral convex cone. Then the following are equivalent:*

(a) $$Ax = b, \qquad x \in S \tag{6}$$

has a solution.

(b) $A^*y \in S^*$ implies $\operatorname{Re} b^*y \geqslant 0$.

THEOREM 3. *Let A, b and S be as in theorem 2. Let $T \subset C^m$ be a polyhedral convex cone. Then the following are equivalent:*

(a) $$Ax - b \in T, \qquad x \in S \tag{7}$$

has a solution.

(b) $A^*y \in S^*, y \in -T^*$ implies $\operatorname{Re} b^*y \geqslant 0$.

Proof. Rewrite (7) as

$$(A, -1)\begin{pmatrix} x \\ z \end{pmatrix} = b, \qquad \begin{pmatrix} x \\ z \end{pmatrix} \in S \times T.$$

By Theorem 2, a solution exists if and only if

$$\begin{pmatrix} A^* \\ -1 \end{pmatrix} y \in (S \times T)^* \qquad \text{implies} \qquad \operatorname{Re} b^*y \geqslant 0.$$

The Theorem follows since $(S \times T)^* = S^* \times T^*$.

COROLLARY 1. *Let $A_1, A_2 \in C^{m\times n}$, $b \in C^m$, let $S_1, S_2 \subset C^n$ be polyhedral convex cones. Then the following are equivalent:*

(a) $$A_1x_1 + A_2x_2 = b; \; A_1{}^*x_2 \in S_2; \qquad x_1 \in S_1 \tag{8}$$

has a solution.

(b) $A_1{}^*y \in S_1{}^*$; $A_2{}^*y + A_1w = 0$, $w \in -S_2{}^*$ implies $\operatorname{Re} b^*y \geqslant 0$.

Proof. Rewrite (8) as

$$\begin{pmatrix} A_1 & A_2 \\ 0 & A_1{}^* \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix} \in \{0\} \times S_2\,; \qquad x_1 \times x_2 \in S_1 \times C^n.$$

By Theorem 3, a solution exists, if and only if,

$$\begin{pmatrix} A_1{}^* & 0 \\ A_2{}^* & A_1 \end{pmatrix}\begin{pmatrix} y \\ w \end{pmatrix} \in (S_1 \times C^n)^*; \qquad -\begin{pmatrix} y \\ w \end{pmatrix} \in (\{0\} \times S_2)^*$$

implies $\operatorname{Re}(b^*y + 0^*w) = \operatorname{Re} b^*y \geqslant 0$.

The Corollary follows, by noting that

$$(S_1 \times C^n)^* = S_1{}^* \times (C^n)^* = S_1{}^* \times \{0\}$$

and

$$(\{0\} \times S_2)^* = \{0\}^* \times S_2{}^* = C^n \times S_2{}^*.$$

THEOREM 4. *Let $A \in C^{m\times n}$, $b \in C^n$, $B \in C^{n\times n}$ be positive semidefinite hermitian, $S \subset C^m$ be a polyhedral convex cone. Then the following are equivalent*:

(a) $\qquad A^*y = Bz + b,\ y \in S^*,\ z^*Bz \leqslant 1,\ Az \in S$

has a solution.

(b) $\qquad Ax \in S$ implies $\operatorname{Re}[(x^*Bx)^{1/2} + b^*x] \geqslant 0$.

Proof. (a) implies (b):

$$x^*A^*y = x^*Bz + x^*b.$$

But

$$\operatorname{Re} x^*Ay = \operatorname{Re} y^*Ax \geqslant 0.$$

Also, $\operatorname{Re} b^*x = \operatorname{Re} x^*b$. Hence, by Lemma 1,

$$\operatorname{Re}[(x^*Bx)^{1/2} + b^*x] \geqslant \operatorname{Re}[(x^*Bx)^{1/2}(z^*Bz)^{1/2} + b^*x] \geqslant \operatorname{Re}[x^*Bz + x^*b] \geqslant 0.$$

(we note for later use that the condition $Az \in S$ in (a) was not used in the above part of the proof.)

(b) implies (a):

Assume (b) holds and (a) does not. Consider the system

$$A^*y - Bz = b,\ y \in S^*,\ Az \in S. \tag{9}$$

If there is no solution, then by Corollary 1, there exists a solution to

$$Ax \in S,\ -Bx + A^*w = 0,\ w \in -S^*,\ \operatorname{Re} b^*x < 0.$$

Now

$$\operatorname{Re} x^*A^*w = \operatorname{Re} w^*Ax \leqslant 0.$$

Also

$$x^*Bx = x^*A^*w.$$

We now have $\operatorname{Re} x^*Bx \leqslant 0$, and, since B is positive semidefinite hermitian, it follows that $x^*Bx = 0$. Hence

$$\operatorname{Re}[(x^*Bx)^{1/2} + b^*x] = \operatorname{Re} b^*x < 0 \qquad \text{contradicting (b).}$$

Thus, (b) implies that (9) has a solution. If, also, $z^*Bz \leqslant 1$, we have (a). Assume, now, that for all (y, z) such that (9) holds, $z^*Bz > 1$.

Consider the quadratic programming problem in complex space

$$\begin{aligned} &\text{Minimize } \tfrac{1}{2}z^*Bz \\ &\text{Subject to } -A^*y + Bz = -b \\ &\qquad\qquad\qquad\quad Az \in S \\ &\qquad\qquad\qquad\quad\ y \in S^*. \end{aligned}$$

The problem is feasible since the constraints are the same as (9), which under our assumptions, has a solution. Since the function to be minimized is bounded below, it follows from a result of Frank and Wolfe [5], that the problem has an optimal solution (y_0, z_0).

Let us rewrite the problem in the form of P1. Thus it becomes

P3. Minimize $\operatorname{Re}\left[\frac{1}{2}\begin{pmatrix} y \\ z \end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + (0)^*\begin{pmatrix} y \\ z \end{pmatrix}\right]$

Subject to $\begin{pmatrix} -A^* & B \\ 0 & A \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} -b \\ 0 \end{pmatrix} \in \{0\} \times S; \qquad \begin{pmatrix} y \\ z \end{pmatrix} \in S^* \times C^n.$

By Theorem 1, its dual is

$$\text{Maximize} \quad \operatorname{Re}\left[-\frac{1}{2}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^*\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} -b \\ 0 \end{pmatrix}\begin{pmatrix} w_3 \\ w_4 \end{pmatrix}\right]$$

$$\text{Subject to} \quad \begin{pmatrix} A & 0 \\ -B & -A^* \end{pmatrix}\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in (S^* \times C^n)^*$$

$$\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \in (\{0\} \times S)^*.$$

This is the same as

$$\begin{aligned} &\text{Maximize } \operatorname{Re}[-\tfrac{1}{2}w_2^*Bw_2 - b^*w_3] \\ &\text{Subject to } Aw_3 \in S,\ -A^*w_4 - Bw_3 + Bw_2 = 0,\ w_4 \in S^*. \end{aligned} \tag{10}$$

By Theorem 1, since there exists an optimal solution (y_0, z_0) of P3, its dual has an optimal solution (w_2, w_3, w_4) with $z_0 = w_2$ and

$$\tfrac{1}{2}z_0{}^*Bz_0 = \text{Re}[-\tfrac{1}{2}w_2{}^*Bw_2 - b^*w_3].$$

Thus,

$$\text{Re} - b^*w_3 = w_2{}^*Bw_2 = z_0{}^*Bz_0 > 1 \tag{11}$$

also

$$(w_2{}^*Bw_2)^{1/2} < w_2{}^*Bw_2 . \tag{12}$$

Now, (10) yields

$$\text{Re}\, w_4{}^*Aw_3 = \text{Re}\, w_3{}^*A^*w_4 \geqslant 0$$

and

$$\text{Re}[w_3{}^*Bw_2 - w_3{}^*Bw_3] = \text{Re}\, w_3{}^*A^*w_4 \geqslant 0. \tag{13}$$

From Lemma 1 and (13) we have

$$w_3{}^*Bw_3 \leqslant \text{Re}\, w_3{}^*Bw_2 \leqslant (w_3{}^*Bw_3)^{1/2}\,(w_2{}^*Bw_2)^{1/2}$$

or

$$(w_3{}^*Bw_3)^{1/2} \leqslant (w_2{}^*Bw_2)^{1/2}. \tag{14}$$

(11), (12) and (14) give

$$\text{Re}[w_2{}^*Bw_2 + b^*w_3] = 0,$$
$$\text{Re}[(w_2{}^*Bw_2)^{1/2} + b^*w_3] < 0,$$

and

$$\text{Re}[(w_3{}^*Bw_3)^{1/2} + b^*w_3] < 0,$$

which, together with $Aw_3 \in S$ of (10), contradicts the assumption (b).

THEOREM 5. *Let A, b, B and S be as in Theorem 4. Then the following are equivalent:*

(a) $$A^*y = Bz + b, y \in S^*, z^*Bz \leqslant 1 \tag{15}$$

has a solution.

(b) $$Ax \in S \text{ implies } \text{Re}[(x^*Bx)^{1/2} + b^*x] \geqslant 0.$$

Proof. (a) implies (b): The proof follows exactly as in Theorem 4.
(b) implies (a): (b) implies the existence of a solution in Theorem 4, (a); such a solution obviously satisfies (15).

SPECIAL CASES

Theorems 4 and 5 yield, as special cases, a number of well-known results.

If $B = 0$, theorem 5 reduces to theorem 2, the extension to complex space of Farkas Lemma [4] given by Ben-Israel [2].

If we choose the cone S as

$$S = \{z \in C^m : |\arg z| \leqslant \alpha\} \qquad \text{for the given} \quad \alpha \in R_+{}^m, \quad \alpha \leqslant \frac{\pi}{2}$$

the polar is easily [2] seen to be

$$S^* = \{w \in C^m : |\arg w| \leqslant \frac{\pi}{2} - \alpha\}$$

(Here, and subsequently, $\pi/2$ means the vector of appropriate dimension with $\pi/2$ in each position). Theorem 4 then gives the equivalence of the following:

$$\text{(a)} \quad A^*y = Bz + b, \quad |\arg y| \leqslant \frac{\pi}{2} - \alpha, \quad z^*Bz \leqslant 1, \quad |\arg Az| \leqslant \alpha \tag{16}$$

has a solution

$$\text{(b)} \qquad |\arg Ax| \leqslant \alpha \text{ implies } \operatorname{Re}[(x^*Bx)^{1/2} + b^*x] \geqslant 0.$$

This is the theorem of Kaul [6]. Theorem 5 gives this result with the condition $|\arg Az| \leqslant \alpha$ eliminated from (16). If also $B = 0$, theorem 5 gives the equivalence of the following:

$$\text{(a)} \qquad A^*y = b, \qquad |\arg y| \leqslant \frac{\pi}{2} - \alpha$$

has a solution.

$$\text{(b)} \qquad |\arg Ax| \leqslant \alpha \text{ implies } \operatorname{Re} b^*x \geqslant 0.$$

This result is an extension of Farkas lemma to complex space given by Levinson [7].

Let all vectors and matrices in theorems 4 and 5 be real. Take $S = S^* = R_+{}^m$. Theorem 4 then gives Eisenberg's result [3], ((1) and (2)). If also $B = 0$, theorem 5 yields Farkas lemma [4], ((3) and (4)).

Remark. Eisenberg's result [3] has also been extended by Sinha [12], Mehndiratta [9] and applied to nondifferentiable programming problems by Sinha [13], Mehndiratta [10], Bhatia and Kaul [14]. Corresponding extensions and applications of our results are possible and will be given in subsequent papers.

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